Eighteen Ways to Attack a Problem

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Difference between university and real-life assignments

When one is taking lecture courses at the university there is plenty of homework and one becomes accustomed to solving problems. However, the problem assignments often differ in a fundamental way from the tasks that are assigned in the outside world or, for that matter, in graduate school. In a homework set one usually knows how to approach the job because, by implication, it has something to do with the lectures. But in real life, which includes research, a task may arise from external demands that do not specify what lecture course, if any, will be relevant. Thus, the first step to take is to decide on a plan of attack.

Need for a variety of tools

This essay is concerned with a variety of approaches that have been found to work. My proposition is that an engineer ought to acquire an armory of assorted tools that can be consciously brought to bear one by one, not too much time being devoted to hammering, if that does not crack the nut, before turning to the saw, and then to the other tools in the toolbox. In order to emphasize some fine distinctions, just for the purpose of the present discussion, I will define some terms as they will be used. Under this terminology, a real-life task is called a puzzle, a term that is chosen to suggest something in common with the puzzles that we are familiar with for entertainment. An *exercise* is a minor variant of an example. A *problem* is a major variant of an example but limited to the same context. A *puzzle* arises when the context is not delimited.

Homework exercises and problems

When you learn about arithmetic progressions the first example that you see is 1 2 3 4… Now here is a typical *exercise* on what you have just learnt: "A series begins 2 4 6 8, what is the next member?" [You verify that the series is compatible with being an AP, determine that the difference is 2, and add 2 to the last member to get the next, which is 10.]

Next is what I would call a *problem*: "A series begins 22 15 8 1, what is the next member?" This problem is excellent; it introduces three new features but lies within the context that has been established. (i) It progresses backwards and thus exhibits to the student that the difference can be negative. (ii) The terms do not have a common factor and thus dispel the possibility, inherent in the previous exercise 2 4 6 8…, that APs in general may be multiples of the original example 1 2 3 4… (iii) The answer is a negative number.

[1] A handout for his course on Two Dimensional Imaging, Winter 1967. Reformated with slight modifications and some [additions in brackets] by J. F. Young, ECE Department, Rice University, 2004. Bracewell is also the author of the indispensable reference *The Fourier Transform and Its Applications* (McGraw-Hill, 1986, ISBN 0-07-007015-6).

Puzzles

The construction of suitable problems is itself in the nature of a puzzle; it is easy to write a homework problem that is like a previous homework problem, but to introduce new features demands creativity. Further excellent problems on APs might move into the complex plane, where circles would be generated, or into three-dimensional space, where helices would appear. But to assign the series 3 15 75 375…, (which is not an arithmetic progression), at this stage without warning, would enlarge the context in an unspecified way and present, in this situation, what I am calling a *puzzle*.

At school we are taught series whose terms are expressible as polynomials and rational functions of polynomials, trigonometric series, the series of prime numbers, and perhaps some others. In the outside world, we encounter series that are offered for entertainment. The format is often intended to mislead us about the context and so the answer is surprising or amusing (at least to the proponent). Here is a famous series of that kind. Given

1 5 10 25 50 100 200 500,

what is the next term? [In case you don't know this one, the following terms are 1,000 2,000 5,000 10,000 100,000 and I think the series stops about there but I am not absolutely sure.]

Although the word puzzle is commonly used for trick questions such as this one, engineering tasks often involve puzzles as defined above in terms of a context that is not delimited. Very often when you are assigned a task, there is no-one to tell you how to do it. People who, at a given stage of their development need to be told what to do, are not the people to whom such tasks should be given (though they often are). It is not uncommon to be given a task that you cannot do or that indeed cannot be done by anyone, because it is impossible. However, what is impossible within one context may be possible in another; the solving of a puzzle often depends on spotting a superior viewpoint. See J. Adams, *Conceptual Blockbusting*, on how conceptual blocks may prevent your progress, and what to do about it. Read about the marvelous problem of how to get a pingpong ball out of a stub of water pipe protruding from the floor of a locked concrete cell when all you and your companions have is a folding carpenter's rule, a hammer and a piece of string.

The following remarks are more or less limited to mathematical aspects rather than to the more general world of engineering. There is a hallway, extremely long and 7 meters wide which takes a sharp 72° turn and continues on for a very long way. What is the longest ladder that can be taken down the hall? This is a good puzzle in solid geometry where the brighter students can be counted on to see that a longer ladder can be taken around the corner if, instead of carrying it horizontally, you slide it along with one end on the floor and the other on the ceiling. Such a student has enlarged his vision to include the third dimension. That is the sort of mathematical context I am going to keep to in this essay, after just one remark. Suppose that an actual ladder of twice that length really does have to be taken down that vast hallway, if possible. Can it be done? In the real world of engineering I would think it probably could, because long ladders are flexible. What's more, they are usually collapsible. At the worst we could saw it in two and reassemble on the other side. When the problem does not specify the set of all acceptable solutions (and it never does in engineering) there is always scope for ingenuity. Even the problem may be challenged. Why use a ladder?

Returning to mathematical topics of the kind we are talking about, we would like to have a stock-in-trade of methods of attack to be applied consciously. Here are some approaches that we have seen in the course on imaging.

1. **When you are asked to prove something, first try to prove it wrong.**
   The reason for this skeptical approach, advocated by Descartes, the inventor of x and y and graphs of algebraic expressions, is that a *single* counter-example suffices to destroy. A single supporting example, or even several, does not prove a proposition (but see footnote to Rule 4; to prove it by the logical standards we learn in mathematics, you must show that it is true for *all* examples, but one counter-example disproves it. Consequently a lot of time can be saved on occasion by a quick disproof.

2. **If it is wrong it may still be useful**
   We saw this with the method of successive substitutions, which may generate a nonconvergent series but even so be useful, as in restoration for running means. In other cases the series may converge, but not in general to the correct result. Yet the limit may still be useful, as when the "principal solution" turns out to be nearly right, as verified by testing.

3. **Make a numerical test.**
   The algebraic formulation of a problem may suggest that you use algebra as a tool, and in consequence you may fill pages with error-ridden symbols, as all of us have done on occasion. To avoid more such debacles, first make a numerical test. You may immediately find the formula wrong, thus making useful progress; but if not, a certain feel for the processes involved is gained that may be helpful in the next attack.

4. **Simplify the problem and solve.**
   Read more about this in Polya, *How to Solve It*. If the problem is two-dimensional and continuous, find the one-dimensional continuous analogue and try to solve. Then solve the two-dimensional discrete. With your mind in the right groove you may now be able to keep all the necessary items in memory at once and solve the full problem.

   Simplify and solve is the single most important piece of advice. If the problem mentions a special function, substitute a Gaussian. If it mentions a Gaussian, change it to an impulse, if it mentions a delta function, change to a rectangle function. Here is a checklist.

Integrate	Sum 2 or 3 values
Differentiate	Subtract adjacent values
N dimensions (or variables)	Try $N-1$, $N-2$, …2, 1
cosine	parabola
delta function	Gaussian
Gaussian	rect x
rect x	$\frac{1}{2}\delta(x+\frac{1}{4})+\frac{1}{2}\delta(x-\frac{1}{4})$
Number[2] > 10	∞

5. **Vary the problem.**
A problem may be couched in intimidating terms. Translate it into familiar ones with with you are more comfortable and where you may possess some intuition. "A bottle and a cork cost tuppence ha'penny and the bottle costs tuppence more than the cork. How much does the cork cost?" This tricky (junior-high-school-level) problem is complicated by archaic terminology that is extinct even in Britain. Translate it into cents.

Translate problems about light into radio, and vice versa. Try water waves, sound waves or sand dunes. Make these variations in your mind as a matter of course when a problem is presented.

6. **Ask someone.**
Some problems appear to have conventional modes of solution but in practice professionals do not approach them the way the books imply. You may think that you can connect your PC to the central computer at work by reading the manuals, but that in fact is not the way experienced people do it. (They may have gained their experience by trying that way.) The correct procedure is to locate someone who already knows. Finding a person is a quite different category of problem, not in itself to be approached lightly. For example, it may do little good to ask either the manufacturer of the PC or the manufacturer of the big computer. The necessary TERMCAP instructions for connection to a VAX may actually reside in ROM in your PC but this vital fact may go unmentioned in the manual and be blanked from the ROM directory.[3] If you wish to identify a tree, which is an important engineering problem, you cannot find out from a botanical key, which is the ostensible procedure. The actual problem is to find someone who knows.

[2] Sometimes numbers greater than 3 are effectively infinite. Indeed, according to a German folk saying "Einmal ist nimmer, zweimal ist immer." That is often a sufficient guide to action. Let *A, B, C, D* be four points on a circle. Then *AB.CD* + *BC.AD* = *AC.BD*. I have found that ordinary Stanford students cannot prove this theorem, but they can take four random points and verify that the two sides are equal to twelve decimals. If you verify to such accuracy three times, how could a theorem with so few degrees of freedom be not true?

[3] The name of this hidden file is /rom/.environ.

7. **Look in a book.**
   If the answer is, or once was, in someone's mind, it may be in a book. If you have to find the impulse response to an electric circuit containing *l* resistors, *m* capacitors and *n* inductances it would be a mistake to approach this by dusting off a text on circuits. All these circuit problems were solved once and forever and the correct answer can be found in P.A. McCollum and B.F. Brown, *Laplace Transform Tables and Theorems* (Holt, Reinhart and Winston, New York, 1965). The solution of differential equations as applied to lumped circuits is a time-honored way of sharpening the mind but is not in use by circuit designers.[4] The same applies to integrals, which should always be looked up in Gradshteyn and Ryzhik, just in case they can be found there. After that, numerical integration may often be satisfactory. Sometimes the sum of very few terms may suffice for an accurate assessment of an integral, as we saw with the computation of Bessel functions from two or three terms (but not terms of the Taylor series!) on a calculator. Acquire experience with how coarse an approximate calculation may be and still be adequate.

8. **Translate into the Fourier domain.**
   Many problems can be transformed into the corresponding problem in another domain and thereby become more readily soluble. Multiplication could be converted to addition by taking logarithms.[5] Convolution becomes multiplication in the Fourier domain. Integration becomes … ahh! A most powerful way of integrating is to go to the Fourier domain F(s), where the central value F(0) will be found to be the desired integral. This devastating simplification is characteristic of the way Fourier domain thinking pays off.

9. **Draw a graph.**
   There are three main modalities for communication: speech, writing and graphics (what else is there?). Nearly everything that can be spoken can be written down (there are exceptions but I cannot give one on this page), and nearly everything that can be written down can be spoken. To some extent conversion to graphics is possible. When you vary the expression of a problem, and change of modality is an extreme case, the problem may seem different and a new angle of attack may suggest itself. This is particularly true when a graph is substituted for an algebraic expression or for a list of data. "Find the area bounded by the curve whose equation $y^2 = (1 - 2|x| + x^2)\,\Pi(x/2)$". It would be a mistake to take the algebraic language of the problem statement as a hint to start integrating symbolically. Better to make a graph and discover that we are looking at the equation of a square! Therefore, the area is 2, just as the area of the circle $y = 1 - x$ is π. You *can* get the answer by integration, but even in the case of $y^2 = 1 - x^2$ I believe you will find, when you begin to integrate, that you are using your geometrical knowledge about the shape of the graph.

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[4] You might also think of enlarging the context and building the circuit.

[5] At one time engineering students used to become so adept at multiplication on the slide rule that their ability to add atrophied. Fortunately there are two ways of adding on a slide rule for use by those who know only how to multiply. The only value remaining for this extinct expertise is to mention it in a footnote as a gesture of solidarity with those who still know what a slide rule is.

10. Make a drawing.

A drawing is not the same as a graph but often a drawing can be made that is equivalent to a graph and yet brings different sorts of intuition into play. Thinking of a function as a brick wall of varying height is reinforced by a three-dimensional isometric drawing that indicates some finite thickness. One cannot think adequately with the expression $z = \sin x$ without drawing or visualizing parallel sand dunes, or the like. The best way to develop ability to visualize in three dimension is to make models; next best is to make drawings. Here is a problem where you had better make a model. "What is the area of the largest flat rigid mat that can be slid around a right-angle bend in a hallway one meter wide, keeping contact with the floor?" Since you may like to try this one I will not divulge the answer but just mention that the area is significantly more than 2 square meters.[6]

11. Try dimensional analysis.

In a vacuum diode the current is proportional to $V^{1.5}$ but a silicon diode involves the logarithm of V and the thermal noise current available from a resistor is proportional to V^{-1}. The flow of current over a weir is rather like that in a vacuum diode, being proportional to the three-halves power of the head. Knowing how such functional dependence originates is worth studying. [See Percy Bridgman, *Dimensional Analysis*, 1963; C. M. Focken, 1953; L.I. Sedov, 1959.]

12. Observe.

The habit of observation is more important in the global context of life than in the restricted context of this essay, but there are exceptions; what is the next symbol in this series, W I T N S? We are *born* with curiosity, but the few people who are observant *acquire* the habit. There are people who see every bird where others see none. Wherever you walk, beetles are more numerous than birds, but a bird watcher may not see beetles. To be observant requires conscious dedication, imagination and experience. An observant mathematician is more likely to be able to continue the above series than a clever one.

13. Experiment.

We are instinctive experimenters in the sense that other mammals experiment in response to innate curiosity, but we are not born with the idea of experiment in the sense made popular by Galileo. I believe, for reasons connected with evolution, that there is survival value in doing what you are told. (Don't go near the river! Don't eat that berry!) Direct and deliberate appeal to Nature through a designed experiment is not yet taught in school; instead, how to appeal to books (i.e. to what is or has been in people's minds) is instilled. Cultivate awareness of the experimental method as a route to results that may or may not be already known. You will need practice, because the design of an experiment, no matter how simple, is one of the highest exercises of the human intellect.

14. Wait for something to turn up.

Since we are genetically disposed to procrastination, there must be survival value in it. If you wait a while, a solution may become apparent, or may become unnecessary. Certainly it is possible to squander much time attacking a problem prematurely.

[6] Since this arrogant challenge was issued in Winter 1987 there has been deathly silence.

However, folk wisdom generally advises against inaction. (A stitch in time saves nine! Don't put off till tomorrow what can be done today! Strike while the iron is hot! There are two extremes to guard against: (a) beating your head against a brick wall, (b) the position of Mr. Micawber, who was always waiting for something to turn up. To avoid being wrecked on Scylla or Charybdis, you must dip a toe in the water today to see if the iron is hot.

15. Let it germinate.
Just as it is good practice to spend the last few minutes of a lecture, in a course of lectures, by starting on the topic of the next lecture, so it is good to start the attack on a problem just as you are finishing a work session. The effect of this is to open registers in the mind that will be needed at the next session. It has often been reported that the subconscious mind produces solutions spontaneously; to take advantage of this phenomenon, bring ingredients of the solution deliberately into consciousness, repeatedly if necessary, to ease the task of the subconscious.

16. Clear the decks for action.
According to Shockley[7], the solution to a given problem may require n concepts to be entertained simultaneously. Since Smith has a limited span of attention, there may not be time for him to summon all n and the given problem may therefore be to him insoluble. The level of difficulty of problems solvable by an individual able to entertain n concepts simultaneously may increase faster than linearly with n (the number of concept-pair interactions goes roughly as n^2, which may be relevant; but how to quantify the "level of difficulty" is not what I am getting at). We are all in the same boat as poor Smith; he can only entertain three ideas at once, I can entertain four, and you may be able to entertain five, but we all have a ceiling. However, if you could raise your ceiling from five to six, an increase of only one, you would raise the concept-pairs from 10 to 15, a fifty per cent jump.

To prepare for such an exertion, purge your short term memory of the irrelevant. One way of doing this consciously is to black outside interruption and engage in some housekeeping activity that does not exercise the mind much but unloads ongoing mental programs. Tidying up your desk is one such preparatory activity; gardening and walking are others.[8] When your attention feels ready to begin a span, start where you left off last time or start solving a related problem (Rules 3 and 4). Since memory does seem to be a limiting factor in deliberately planned sessions of this kind, make notes, before terminating, intended for yourself to read at the next session in order to come up to speed with the least erosion of your attention span.

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[7] [William Shockley, Stanford EE Professor, inventor of the transistor, Nobel Prize winner.]
[8] Since writing this I found that Atanasoff cleared his mind for the invention of the logic system for the first electronic computer in 1935 by driving 200 miles through the night (*Physics Today*).

17. Look in the Encyclopedia Britannica.
A good deal of what you need to know as background is in the *Encyc. Brit.*, with one general exception: it is not a good place to look for information on plumbing, carpentry or bricklaying. Learn how to use this indispensable single reference.

18. Siegman's[9] method.
Give it to someone else to solve. This reminds me of what mad King Ludwig said, "Ich werde darüber nachdenken laßen."

[9] [A. E. Siegman, Professor of EE, Stanford University, author of *Lasers* (University Science Books, 1986).]